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1983 J. Phys. A: Math. Gen. 16 4417

(http://iopscience.iop.org/0305-4470/16/18/040)

J. Phys. A: Math. Gen. 16 (1983) 4417–4421. Printed in Great Britain

COMMENT

Comments on constraints of gauge theories

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Received 4 July 1983

Abstract. Due to an improper interpretation of the relation of first-class constraints to gauge degrees of freedom, many authors have asserted that all primary and secondary first-class constraints are associated with gauge freedom and enter in the generator of the evolution of the system. By presenting several examples we show that secondary first-class constraints are not associated with gauge degrees of freedom, but only intrinsic first-class constraints yield gauge freedom.

In previous papers (Sugano and Kamo 1982, Sugano 1982, Sugano and Kimura 1983a, b), we investigated dynamical systems with constraints and showed that secondary first-class constraints (FCC) are not necessarily associated with gauge degrees of freedom, but only intrinsic FCCs yield gauge freedom. Our argument was based on the consistency between the velocity phase space formalism and the phase space one, or a consistent quantisation by means of the Dirac formalism. The intrinsic constraints Φ_α are defined by the constraints which are related to the kernels τ^i_α of the Hessian matrix $A_{ij} \equiv \partial^2 L/\partial \dot{q}^i\, \partial \dot{q}^j$ by (Sugano and Kamo 1982)

$$\tau^i_\alpha \simeq \partial\Phi_\alpha/\partial p_i \qquad (\alpha = 1, \ldots, r;\ i = 1, \ldots, n) \tag{1}$$

$$A_{ij}\tau^j_\alpha \simeq 0 \tag{2}$$

where p_i are momenta conjugate to q^i and the rank of A_{ij} is supposed to be $n-r$. The symbol $\simeq$ means identity or weak equality ($\approx$), that is, equations (1) and (2) are identities for primary constraints, while being weak equalities for secondary ones. The latter case occurs when the rank of A_{ij} is reduced by constraints.

In order to quantise such a dynamical system with FCCs by means of the Dirac bracket method, we must impose gauge conditions to turn all FCCs into second class constraints (SCC) and to pick up physical variables. Since the number of gauge constraints which can be arbitrarily chosen is not more than the number of gauge degrees of freedom (i.e. the number of intrinsic FCCs), non-intrinsic FCCs usually remain intact as FCCs. To convert the non-intrinsic FCCs into SCCs consistently we proposed two methods (Sugano and Kimura 1983a, b). One is to employ, on the constraint submanifold, stationary constraints independent of the gauge conditions. Such stationary constraints are essentially equivalent to first integrals of equations of motion of the system, so that they are not external constraints. An alternative method is to choose gauge conditions such that the required number of descendant constraints follows by repeating, the same times as the number of the secondary FCCs, the algorithm with respect to stationarity of the gauge constraints and gauge functions v in the total Hamiltonian are fixed in the final step of the algorithm. The existence of such gauge conditions was proved (Sugano and Kimura 1983b), provided that a further condition such as Lorentz covariance was not imposed.

Due to an improper interpretation of the relations of the gauge degrees of freedom to the FCCs, there have still appeared arguments asserting that all primary and secondary FCCs are associated with gauge freedom (Gotay 1983, Di Stefano 1983, Appleby 1982). The aim of this paper is to repeat again our proposition presented in the previous papers by showing further examples (especially in connection with the examples given by Gotay (1983)).

Let us first consider the case of electromagnetic field. From the Lagrangian density

$$L=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad \text{with } F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu \tag{3}$$

we obtain the canonical Hamiltonian density

$$H=\tfrac{1}{2}\pi_k\pi^k+\pi^k\,\partial_k A_0+\tfrac{1}{4}F_{ij}F^{ij} \tag{4}$$

and the primary and secondary FCCs

$$\Phi=\pi^0\approx 0 \qquad \chi=\partial_k\pi^k\approx 0 \tag{4a, b}$$

where π^μ are momenta conjugate to A_μ.

If the extended Hamiltonian density

$$H_{\mathrm{E}}=H+v_0\pi^0+v_1\,\partial_k\pi^k=H_{\mathrm{T}}+v_1\,\partial_k\pi^k \tag{5}$$

is employed as the generator of evolution of the system and Φ and χ are associated with the gauge degrees of freedom, we shall be led to inconsistent results. Since two independent gauge conditions may be arbitrarily chosen in this case, let us take the gauges

$$\Psi^0=A_0\approx 0 \qquad \Psi^1=A_3\approx 0. \tag{6}$$

Then we obtain from the stationarity condition of Ψ^0 and Ψ^1

$$v_0=0 \qquad v_1=\partial_3^{-1}\pi^3 \tag{7}$$

and

$$\pi_3=\partial_3 A_0-\partial_0 A_3\approx 0. \tag{8}$$

From $\dot{\pi}_3\approx 0$ and the equations of motion

$$\dot{\pi}^k=\partial_i F^{ik}$$

it follows that

$$\partial_3(\partial_1 A^1+\partial_2 A^2)\approx 0. \tag{9}$$

So, we have only one independent component A_2 (or A_1) for a wave propagating in the x_1–x_3 plane (or the x_2–x_3 plane).

This inappropriate result is due to imposing two independent gauge conditions.

On the other hand, instead of the Lagrangian density (3), Gotay (1983) took a modified one

$$L=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}\lambda^{-1}(\partial_\mu A^\mu)^2 \tag{10}$$

where λ is a Lagrange multiplier. Then we find

$$\Phi = \pi^\lambda \approx 0 \tag{11}$$

and

$$H_T = \tfrac{1}{2}\pi_k\pi^k + \pi^k\,\partial_k A^0 + \tfrac{1}{2}\lambda(\pi^0)^2 + \pi^0\,\partial_k A^k + \tfrac{1}{4}F_{ij}F^{ij} + v_0\pi^\lambda. \tag{12}$$

Secondary constraints are

$$\chi_1 = \pi^0 \approx 0 \qquad \chi_2 = \partial_k\pi^k \approx 0. \tag{13}$$

Φ, χ_1 and χ_2 are FCCS so that the extended Hamiltonian density is

$$H_E = H_T = v_1\pi^0 + v_2\,\partial_k\pi^k. \tag{14}$$

Now choosing the three gauge conditions

$$\Psi^0 = \lambda \approx 0 \qquad \Psi^1 = A_0 \approx 0 \qquad \Psi^2 = A_3 \approx 0 \tag{15}$$

we obtain again (9).

It should be noticed, apart from the gauge conditions, that the definition of momenta

$$\pi^0 = \lambda^{-1}(\dot{A}_0 - \partial_k A^k) \qquad \pi^k = \dot{A}^k - \partial^k A_0 \tag{16}$$

and the equations of motion derived from H_E

$$\dot{A}_0 = \partial_k A^k + v_1 \qquad \dot{A}_k = \pi_k + \partial_k A_0 - \partial_k v_2 \tag{17}$$

yield $v_1 \approx v_2 \approx 0$, owing to the constraints (13). A similar argument for the ordinary Lagrangian density (3) and H_E of (5) leads to $v_1 = 0$. It means that the Hamiltonian consistent to the Lagrangian formalism is not H_E but $H_T = H + v_0\Phi$.

In connection with FCCS, we next comment on degrees of freedom of a dynamical system. In order to eliminate a degree of freedom, two constraints are in general needed, irrespective of any class of constraints. For primary FCCS Φ_A in the phase space, gauge conditions Ψ_A^0 are imposed to eliminate the corresponding gauge variables. For secondary FCCS χ_A^k where

$$\chi_A^k \approx \{\chi_A^{k-1}, H_T\} \qquad (k = 1, \ldots, m_A) \tag{18}$$

the corresponding degrees of freedom should be eliminated by imposing constraints Ψ_A^k. Here all Ψ_A^k must be stationary on the constraint submanifold. Then Ψ_A^k except Ψ_A^0 are not arbitrary but essentially first integrals of equations of motion (note that the stationary constraints Φ_A and χ_A^k are essentially a sort of first integrals whose integration constants are fixed to particular values). A consistent set of Ψ_A^k is obtained successively by (Sugano and Kimura 1983b)

$$\Psi_A^k \approx \{\Psi_A^{k-1}, H_T\} \qquad (k = 1, \ldots, m_A) \tag{19}$$

with the conditions

$$\begin{aligned} &\{\Phi_A, \Psi_{A'}^l\} \approx 0 \qquad (\text{for } l < m_{A'}) \\ &\{\Phi_A, \Psi_A^{m_A}\} \not\approx 0. \end{aligned} \tag{20}$$

For the Lagrangian density (3) for the electromagnetic fields, the FCCS are given by (4*a*, *b*). The Coulomb gauge

$$\Psi^0 = \partial_i A^i \approx 0 \tag{21a}$$

yields

$$\dot{\Psi}^0 \approx \Delta A_0 \approx 0 \rightarrow \Psi^1 = A_0 \approx 0 \tag{21b}$$

and by $\dot{\Psi}^1 \approx 0$, the gauge function v_0 in $H_T = H + v_0\pi^0$ is fixed to be zero. Φ, χ, Ψ^0 and Ψ^1 form a set of stationary SCCs by which two unphysical components are eliminated.

We note that the generator of the gauge transformation for the Lagrangian density (3) is given by

$$G = \int \mathrm{d}^4x\,[\dot{\varepsilon}(x)\Phi - \varepsilon(x)\chi] = \int \mathrm{d}^4x\,\partial_\mu\varepsilon(x)\cdot\pi^\mu(x). \tag{22}$$

Next it should be noted that the temporal gauge $A_0 \approx 0$ without taking account of $\partial_k A^k \approx 0$ is not enough to eliminate the two unphysical components, even if physical states are restricted by

$$\chi|\text{phys}\rangle = \partial_k\pi^k|\text{phys}\rangle = 0. \tag{23}$$

As is well known, an inconsistent result is derived from (23). A consistent constraint is given by

$$(\partial_k\pi^k)^+|\text{phys}\rangle = 0 \tag{24}$$

where the positive frequency part of $\partial_k\pi^k$ can be expressed in the momentum space in terms of $\chi = \partial_k\pi^k$ and $\Psi^1 = \partial_k A^k$ for the temporal gauge:

$$\chi^+(\boldsymbol{p}) = (2\pi)^{-3/2}\int \mathrm{d}^3x\,\mathrm{e}^{-\mathrm{i}\boldsymbol{px}}\sqrt{p_0/2}[\chi(0,\boldsymbol{x}) - \mathrm{i}p_0\Psi^1(0,\boldsymbol{x})] \tag{25}$$

owing to χ and Ψ^1 being time independent. This equation shows that both $\chi \approx 0$ and $\Psi^1 \approx 0$ should be taken into account in order to guarantee (24). From the above argument, it will be not difficult to see why the quantisation of the Yang–Mills fields in the temporal gauge with the restriction

$$(\partial_k + g\boldsymbol{A}_k \times)\boldsymbol{\pi}^k|\text{phys}\rangle = 0$$

is questionable (Kakudo *et al* 1983). If it is assumed that $g \rightarrow 0$ in asymptotic states, we have the same situation as for the electromagnetic fields.

Gotay's interpretation (Gotay 1983) of degrees of freedom concerning the Lagrangian

$$L = (1/2x)\dot{y}^2 \tag{26}$$

is also inadequate. In the phase space we have

$$H_T = \tfrac{1}{2}xp_y^2 + vp_x \tag{27}$$

and

$$\Phi = p_x \qquad \chi = p_y. \tag{28}$$

Then p_x and p_y are not true dynamical variables. Now putting

$$\Psi^0 = y + c \tag{29}$$

we obtain

$$\{\Psi^0, H_T\} = xp_y \approx 0$$

namely Ψ^0 is a constant of motion. Hence Ψ^1 may be arbitrarily chosen:

$$\Psi^1 = x + c' \tag{30}$$

and then $\{\Psi^1, H_T\} = v = 0$. The Φ, χ, Ψ^0 and Ψ^1 are SCCs and reduce the phase space to one point.

This system has only one gauge degree of freedom for an x transformation, as is easily seen in the Lagrange formalism. The generator of the gauge transformation (Sugano and Kamo 1982) is

$$G = \dot{\varepsilon}(t)p_x - \tfrac{1}{2}\varepsilon(t)p_y^2 \approx \dot{\varepsilon}p_x. \tag{31}$$

Gotay did not think of imposing another stationary constraint conjugate to the secondary FCC χ. So he asserted that the system has $\frac{1}{2}$ degrees of freedom when $\chi = p_y$ is not associated with gauge freedom.

The Lagrangian (26) is a special example. A more proper example illustrating this circumstance was presented in our previous paper (Sugano and Kimura 1983a).

Consequently the secondary FCCs are not associated with gauge degrees of freedom but only the intrinsic FCCs yield gauge freedom. For the secondary FCCs, we should impose stationary (not arbitrary) constraints or descendant constraints which follow from arbitrary gauge conditions.

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